What is Voltage anyway?

1 Electrostatics

Electrostatics can be summed up by Coulomb's law and the law of superposition.

Coulomb's law determines the force between two fixed charges in space:

$$\mathbf{F_{1,2}} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}} \tag{1}$$

where $\mathbf{F_{1,2}}$ denotes the force on charge q_1 due to charge q_2, r denotes the distance between the two charges, and $\hat{\mathbf{r}}$ denotes the unit vector in the direction from charge q_2 to q_1.

You know the deal with superposition. If there are a bunch of charges in space (say, N of them), then the net force felt by another charge Q is just the sum of all the individual forces charge Q would feel with each of the N charges alone:

$$\begin{aligned} \mathbf{F_Q} &= \sum_{i=1}^{N} \frac{1}{4\pi\epsilon_0}\frac{Q q_i}{r_i^2}\hat{\mathbf{r}}_\mathbf{i} \\ &= \frac{Q}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{\mathbf{r}}_\mathbf{i} \end{aligned} \tag{2}$$

where $\mathbf{r_i}$ denotes the vector from the location of q_i to that of Q. Many times, people refer to Q as a "test charge," because we're really asking, "Suppose I put a charge Q in this location. What force would it feel?"

BTW, we've been talking about discrete point charges here. If you want to get fancy and talk about continuous charge distributions, then the summation in equation 2 can be expressed as a volume integral containing that "charge density" buzz word:

$$\mathbf{F_Q} = \frac{Q}{4\pi\epsilon_0}\int_{\text{all space}}\frac{\rho}{r^2}\hat{\mathbf{r}}\,d\tau \tag{3}$$

where ρ specifies the charge density at each point in space (in Coulombs per meter, or whatever).

Whether you prefer to use equation 2 or 3, the required information is the same for both. You must specify the following three things:

1. The charge distribution: In the case of equation 2, this means you specify a table with N rows and 4 columns. One row for each charge. And the columns are charge (in Coulombs) and the x, y and z coordinate of the charge. If using equation 3, you specify the charge density function $\rho(x, y, z)$.
2. The location of the test charge
3. The charge of the test charge itself (in Coulombs)

OK, so that's all there is to electrostatics. You tell me where and how big the charges are, and equation 2 tells you the force that a charge Q will feel if placed at some specified location. That's kinda what you're after when you do your, "So, I'm an electron" spiel, right?

So since force is such a natural thing to think about, why do we always talk about Volts, or this weird $\mathbf{E}$ field thing?

2 The Electric Field

Let's start with the **E** thing. How about this statement: I bet you that if you were in the business of electrostatics, and the only tool in your belt was equation 2, you'd eventually come up with **E** on your own (OK, maybe you would pick a different letter). Here's why. A typical day would consist of some dude coming to you with a paper or CSV file containing a big list or table describing a charge distribution (or charge density function) and also the coordinates and charge of a test charge. And he'd ask you to figure out the force on his test charge. And you'd calculate it using equation 2 (or 3) and give him an answer. Then he'd knock on the door at 5am the next morning and ask, "Damn, what if my test charge is twice as big? Are you gonna have to re-crunch all that data?" And you'd look at equation 2, and be like, "No dipshit, can't you see that the force just doubles if you double Q?" And after this happened to you a few hundred times, you'd get fed up. And you'd just divide both sides of equations 2 and 3 by Q, and report that as your solution to your dipshit customers. You'd tell them, "Look, I don't care how big your test charge is, just tell me *where* it is and gimme your charge distribution, and that's all I need. Then take the solution I give you, multiply it by the size of your test charge, and that's the force your test charge will feel." This new thing (BTW it's still a vector just like the force was) you're giving your customers is just the force-per-unit-charge that a test charge would feel. And this thing is called the electric field:

$$\mathbf{E} = \frac{\mathbf{F_Q}}{Q} \tag{4}$$

If this is totally obvious and/or insulting and/or boring, skip to next section. If not, realize that we do this type of thing all the time. For example, if you're comfortable with the gravitational field constant, $g \approx 9.81\text{m/s}^2$, I argue you then have to be comfortable with the electric field. It's exactly the same idea. If someone asks us what the gravitational force is on an object near the earth, we just tell them to multiply g by the mass of the object.

Just to pump of my word count, let's explicitly write the expression for the electric field due to a system of point charges (from equation 2):

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{\mathbf{r}}_\mathbf{i} \tag{5}$$

And similarly for a continuous charge distribution:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int\limits_{\text{all space}} \frac{\rho}{r^2} \hat{\mathbf{r}} d\tau \tag{6}$$

One final note here: It's important to realize that while we cleverly dumped the requirement of specifying the value of the test charge, Q, we did not rid ourselves of needing to specify its (or at least some) location. Recall that the $\mathbf{r}$ vectors above represent the difference between this location of interest and a point in the charge distribution. This is a painfully long way of saying that for a given charge distribution, the electric field is a function of position: $\mathbf{E}(\mathbf{x})$.

3 Voltage (electric potential)

You might be thinking that section 1 was all you needed to know how an electron feels, and you're right. But hopefully, section 2 convinced you there's a slight benefit in dealing with **E** instead of $\mathbf{F_Q}$. Lemme just cut to the chase and smack you with the formal definition of voltage (or electric potential), and then I'll try to justify why we use it later.

3.1 Definition

The definition of voltage can be cast in two forms - both of which are equivalent.[1] The first form is:

$$V(\mathbf{b}) - V(\mathbf{a}) = -\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot \mathbf{d}\ell \tag{7}$$

where the integral is a path integral from location **a** to location **b**. The second form is:

$$\mathbf{E} = -\nabla V \tag{8}$$

3.2 But what does it mean?

Equation 7 tells you how to determine the voltage change from one spot to another if you know the electric field. One cool way to look at it is in terms of energy. Plugging equation 4 into equation 7, we get:

$$V(\mathbf{b}) - V(\mathbf{a}) = -\frac{1}{Q}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F_Q} \cdot \mathbf{d}\ell = \frac{W_{Q,\mathbf{a}\to\mathbf{b}}}{Q} \tag{9}$$

where the path integral now looks like (and is) a work integral. Recall that $\mathbf{F_Q}$ denotes the net force on a test charge Q due to some charge distribution. So if you wanted to hold the test charge still, you'd have to apply (with your fingers of course) an equal and opposite force. And if you wanted to to push this test charge very slowly from a point **a** to another point **b** in the presence of other charges (or an electric field), you'd have to apply a force infinitesimally larger in magnitude and opposite in sign to whatever $\mathbf{F_Q}$ is at every point along the way. And therefore, the quantity $W_{Q,\mathbf{a}\to\mathbf{b}}$ in equation 9 represents the work that would be required by you or someone else to move a test charge Q from **a** to **b**. So sometimes, it's useful to think of voltage as "work per unit charge."

3.3 Why bother?

The second definition of voltage (equation 8) is useful if the voltage is known and the electric field is desired. Just take the gradient of the voltage, negate it, and there you are. You might be thinking, "But all I know from electrostatics is how to determine **E** from a charge distribution. Why would I know V instead?" I can think of two answers to that one:

1. I didn't describe them, but trust me when I tell you there are ways to determine electric potential directly from a charge distribution. And not only are there ways, but there are also reasons. For example, the potential of a point charge q is given by: $V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$, where r again denotes the distance between the point charge and the location of interest (i.e. where a test charge would be placed). Once again, superposition applies to voltage just as it did to force, and so you can image what the resulting summation or integral would look like for a distribution of charge. So why chose to do it this way as opposed to just working with **E**? It's a matter of convenience and practicality. Firstly, V is a scalar quantity, while **E** is a vector. And secondly, thinking about the continuous case, integrating something with $\frac{1}{r}$ is twice as easy as an integrand with $\frac{1}{r^2}$. OK, maybe not twice, but a lot.

2. In most of the labs I've worked in, power supplies and DMM's read out in volts.

[1] Actually, that's not *exactly* true. I'll resist the urge to spill on this one, but ask me if you ever have 20 minutes that you're willing to never get back.

3.4 Simplifying (for managers only)

In case some or all knowledge of the vector calculus above has been deleted from memory in favor of PowerPoint keyboard shortcuts, try this out. If we boil everything down to the 1-D case, equations 7 and 8 reduce to:

$$V(b) - V(a) = -\int_a^b E(x)\, dx \tag{10}$$

and

$$E(x) = -\frac{dV}{dx} \tag{11}$$

Bottom line, the electric field (and thus the force per unit test charge) is just the spatial derivative of voltage.

4 Relating it all to circuits

When it comes to electrical circuits, we spend our time talking about Volts and Amps. Wise men hand us little formulas that relate the two for various circuit elements (resistors, caps, inductors, etc.). Kirchoff gift wraps two little laws for us (KVL and KCL). And from that point on, it's just math. Unless you're the type who needs to know how the electrons feel. You know, an electron shrink. I'll leave the comprehensive circoanalysis for him, but will at least try to break down all the piece parts.

4.1 Voltage

See section 3!

4.2 Current

You know, just count how much charge passes a certain point in the circuit per unit time.

For a particular material, there's a certain amount of free charge (electrons) available to wonder about. And in a nice thin wire, we can imagine talking about this in terms of a linear charge density, λ, given in Coulombs per unit length. If this linear charge distribution moves with speed v, then in a time interval Δt, an amount of charge equal to $\lambda \cdot v \cdot \Delta t$ will pass through each point in the wire. This implies:

$$i = \lambda v \tag{12}$$

which let's us think of current as the speed at which charge is flowing. Of course, we know that at the lowest level, all the lil' electrons are bouncing around like crazy almost at the speed of light, and in all directions. However, equation 12 still applies on average, and v represents what we call the "drift velocity" of the current.

4.3 Kirchoff's Voltage Law (KVL)

Sum all the voltage deltas around any closed loop in a circuit, and you better get zero. Might sound obvious.

4.4 Kirchoff's Current Law (KCL)

Sum all currents into a node (or junction) of a circuit, and you better get zero. Probably seems real obvious. Unless you consider the chance of charge piling up (accumulating) in that junction, in which case KCL is wrong. Luckily that's pretty rare.

4.5 Resistors

So we all know that $V = iR$, right? And by the way, where's it come from? You can derive it from Maxwell's equations of course (I can hear him saying it with that smooth accent). Ooh, sorry, but thanks for playing. While resistance is the easiest thing to deal with at the circuit level, it's actually more complicated than capacitance and inductance at the lower level. And it has nothing to do with Maxwell's equations. Fundamentally, Ohm's Law is its own empirical law. Ohm just noticed that in most materials, current flow is proportional to the electric field,**E**. We can think of current as the relative speed of the flowing electrons (see equation 12). And we know from above (equation 4) that **E** is really just like force (per unit charge). So I'm basically saying that force is proportional to speed. Or at least that's what Ohm says. But what about Newton? I thought force was supposed to be proportional to acceleration - not speed. OK, so they're both right. Electrical resistance is really a sort of viscous drag force on the electrons as they bounce around like little pinballs. The faster they flow (or drift), the more drag there is due to them bumping into the crystal lattice. Just like the case of aerodynamic drag, where drag force increases with speed. But unlike aero drag, which is not very well proportional to speed,[2] the electrical drag is almost perfectly proportional to drift speed for most materials. Lucky us.

Putting it all into equation format, the low-level version of Ohm's Law in a wire (1-D) is given by:

$$iR' = E \tag{13}$$

where R' denotes the electrical resistance per unit length of the wire, and E (now written as a scalar) denotes the component of **E** along the length of the wire. And now putting equation 10 to good use, we can just integrate equation 13 along a length L of the wire to arrive at:

$$\Delta V = -i \underbrace{R'L}_{R} \tag{14}$$

which states that the voltage will drops as current flows through a resistor. But we already knew that.

4.6 Capacitors

I'm sure you remember the model we use for a capacitor. Take two parallel plates each with area A and separated by a distance d. Then cram a certain amount, Q, of positive charge on one plate[3], and cram an equal amount of negative charge on the other. With this problem statement along with some electrostatics tools in our belt (remember Gauss's law?), it's not difficult to determine that what results is an approximately uniform **E** field between the two plates. The component of the field directed from one plate to the other is given by:

$$E = \frac{Q}{\epsilon_0 A} \tag{15}$$

And since voltage is just the integral of electric field, the voltage ramps linearly from one plate to the other:

$$V_{pos} - V_{neg} = Ed = \frac{dQ}{\epsilon_0 A} \tag{16}$$

Convention has us labelling the quantity $\frac{\epsilon_0 A}{d}$ as the capacitance, C. So we can express the voltage across a capacitor as:

$$V_{cap} = \frac{Q}{C} \tag{17}$$

In terms of electron shrinkery, the electrons on the negative plate are very attracted to the positively charged plate, and this is a very real force that you could measure between the two plates if you had a load cell. I think I can do better in the shrinkery department regarding capacitors, but sick of typing now, so will have to wait...

[2] Depending on a bunch of factors, aerodynamic drag is often closer to scaling with the square of speed.
[3] As you know, we do this by sucking out electrons

4.7 Inductors

Don't think you care about these right now, so you'll have to wait...