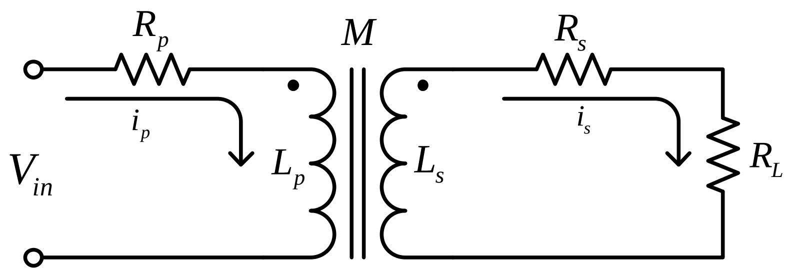
Simple Transformer Analysis
7-28-08



KVL on the primary side gives:

$$V_{in} = R_p i_p + \varepsilon_p \tag{1.1}$$

where the primary winding EMF is given by (see appendix A):

$$\varepsilon_p = L_p \frac{di_p}{dt} - M \frac{di_s}{dt} \tag{1.2}$$

KVL on the secondary side gives:

$$\varepsilon_s = \left(R_s + R_L\right) i_s \tag{1.3}$$

where the secondary EMF is given by:

$$\varepsilon_s = M \frac{di_p}{dt} - L_s \frac{di_s}{dt} \tag{1.4}$$

Combining (1.1) and (1.2):

$$V_{in} = R_p i_p + L_p \frac{di_p}{dt} - M \frac{di_s}{dt} \tag{1.5}$$

Combining (1.3) and (1.4):

$$M \frac{di_p}{dt} - L_s \frac{di_s}{dt} = \left(R_s + R_L\right) i_s \tag{1.6}$$

Note that the mutual inductance is given by:

$$M = \sqrt{L_p L_s} \tag{1.7}$$

Rewriting the winding EMF's:

$$\varepsilon_p = L_p \frac{di_p}{dt} - \sqrt{L_p L_s} \frac{di_s}{dt} \tag{1.8}$$

$$\varepsilon_s = \sqrt{L_p L_s} \frac{di_p}{dt} - L_s \frac{di_s}{dt} \tag{1.9}$$

Note that:

$$\sqrt{\frac{L_s}{L_p}} \cdot \varepsilon_p = \sqrt{L_p L_s} \frac{di_p}{dt} - L_s \frac{di_s}{dt} \Rightarrow \varepsilon_s = \sqrt{\frac{L_s}{L_p}} \cdot \varepsilon_p \tag{1.10}$$

In terms of number of turns in each of the windings, this becomes:

$$\boxed{\varepsilon_s = \frac{N_s}{N_p}\varepsilon_p} \tag{1.11}$$

Plugging (1.11) into (1.1), we get:

$$V_{in} = R_p i_p + \frac{N_p}{N_s}\varepsilon_s \tag{1.12}$$

And plugging (1.3) into this, we get:

$$V_{in} = R_p i_p + \frac{N_p}{N_s}\left(R_s + R_L\right) i_s \tag{1.13}$$

Taking the Laplace transform of (1.5) and (1.6), we get:

$$V_{in} = \left(L_p s + R_p\right) i_p - \left(Ms\right) i_s \tag{1.14}$$

and

$$\left(Ms\right) i_p = \left[L_s s + \left(R_s + R_L\right)\right] i_s \tag{1.15}$$

Plugging (1.15) into (1.14) to eliminate i_s :

$$\begin{aligned} V_{in} &= \left[\left(L_p s + R_p\right) - \frac{M^2 s^2}{L_s s + \left(R_s + R_L\right)}\right] i_p \\ &= \frac{\left(L_p s + R_p\right)\left[L_s s + \left(R_s + R_L\right)\right] - M^2 s^2}{L_s s + \left(R_s + R_L\right)} i_p \\ &= \frac{\left(L_p L_s - M^2\right) s^2 + \left[L_p\left(R_s + R_L\right) + L_s R_p\right] s + \left(R_s + R_L\right) R_p}{L_s s + \left(R_s + R_L\right)} i_p \\ &= \frac{\left[L_p\left(R_s + R_L\right) + L_s R_p\right] s + \left(R_s + R_L\right) R_p}{L_s s + \left(R_s + R_L\right)} i_p \end{aligned} \tag{1.16}$$

The effective impedance of the circuit as seen by the input voltage source is:

$$\boxed{Z_{eff}\left(s\right) = \frac{V_{in}\left(s\right)}{i_p\left(s\right)} = \frac{\left[L_p\left(R_s + R_L\right) + L_s R_p\right] s + \left(R_s + R_L\right) R_p}{L_s s + \left(R_s + R_L\right)}} \tag{1.17}$$

Looking at this TF for extremely low and high frequencies:

$$\lim_{s \to 0} Z_{eff}\left(s\right) = R_p \tag{1.18}$$

and

$$\begin{aligned} \lim_{s \to \infty} Z_{eff}\left(s\right) &= \frac{L_p\left(R_s + R_L\right) + L_s R_p}{L_s} \\ &= \frac{L_p}{L_s}\left(R_s + R_L\right) + R_p \\ &= \frac{N_p^2}{N_s^2}\left(R_s + R_L\right) + R_p \end{aligned} \tag{1.19}$$

The total flux in the core is given by:

$$\Phi = \Phi_p + \Phi_s \tag{1.20}$$

where:

$$\Phi_p = \frac{\mu A_c N_p}{\ell_m} i_p \tag{1.21}$$

and

$$\Phi_s = -\frac{\mu A_c N_s}{\ell_m} i_s \tag{1.22}$$

The winding EMF's are given by:

$$\varepsilon_p = N_p \frac{d\Phi}{dt} = N_p \left(\frac{d\Phi_p}{dt} + \frac{d\Phi_s}{dt} \right) = \underbrace{\frac{\mu A_c N_p^2}{\ell_m}}_{L_p} \frac{di_p}{dt} - \underbrace{\frac{\mu A_c N_p N_s}{\ell_m}}_{M_{ps}} \frac{di_s}{dt} \tag{1.23}$$

$$\varepsilon_s = N_s \frac{d\Phi}{dt} = N_s \left(\frac{d\Phi_p}{dt} + \frac{d\Phi_s}{dt} \right) = \underbrace{\frac{\mu A_c N_p N_s}{\ell_m}}_{M_{sp}} \frac{di_p}{dt} - \underbrace{\frac{\mu A_c N_s^2}{\ell_m}}_{L_s} \frac{di_s}{dt} \tag{1.24}$$

Note that:

$$M_{ps} = M_{sp} = M \tag{1.25}$$

Rewriting the winding EMF's:

$$\varepsilon_p = L_p \frac{di_p}{dt} - M \frac{di_s}{dt} \tag{1.26}$$

$$\varepsilon_s = M \frac{di_p}{dt} - L_s \frac{di_s}{dt} \tag{1.27}$$