Assume we have an N^{th} order transfer function,

$$G(s)=\frac{Y(s)}{X(s)}=\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \qquad \{1\}$$

Assuming a sinusoidal input with amplitude *A* and frequency ω,

$$x(t)=A\sin(\omega t) \;\Rightarrow\; X(s)=\frac{A\omega}{s^2+\omega^2},$$

the output becomes,

$$Y(s)=\frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}\cdot\frac{A\omega}{s^2+\omega^2}$$

Doing PFE,

$$Y(s)=\frac{r_1}{(s-p_1)}+\frac{r_2}{(s-p_2)}+\cdots+\frac{r_n}{(s-p_n)}+\frac{as+b}{s^2+\omega^2} \qquad \{2\}$$

where,

$$\begin{aligned} r_k &= \left[Y(s)\cdot(s-p_k)\right]\Big|_{s=p_k} \\ &= \frac{A\omega(p_k-z_1)(p_k-z_2)\cdots(p_k-z_m)}{(p_k-p_1)\cdots(p_k-p_{k-1})(p_k-p_{k+1})\cdots(p_k-p_n)(p_k^2+\omega^2)} \end{aligned} \qquad \{3\}$$

and

$$ja\omega+b=\left[Y(s)\cdot(s^2+\omega^2)\right]\Big|_{s=j\omega}=A\omega\frac{(j\omega-z_1)(j\omega-z_2)\cdots(j\omega-z_m)}{(j\omega-p_1)(j\omega-p_2)\cdots(j\omega-p_n)} \qquad \{4\}$$

Comparing the last term to {1}, we can write:

$$ja\omega+b=A\omega\cdot G(j\omega) \qquad \{5\}$$

where $G(j\omega)$ is the transfer function with *s* replaced by $j\omega$, and is called the sinusoidal transfer function. Equating the real and imaginary components of {5}, we can solver for *g* and *h*:

$$b=A\omega\cdot\text{Re}\left\{G(j\omega)\right\} \qquad \{6\}$$

$$a=A\cdot\text{Im}\left\{G(j\omega)\right\} \qquad \{7\}$$

Performing the ILT on {2},

$$y(t)=r_1e^{p_1t}+r_2e^{p_2t}+\cdots+r_ne^{p_nt}+\frac{b}{\omega}\sin(\omega t)+a\cos(\omega t) \qquad \{8\}$$

If $p_k<0$ for k=1 thru *n*, then all but the last two terms in {8} will approach zero exponentially as time increases. Therefore,

$$y_{ss}(t)\equiv\lim_{t\to\infty}y(t)=\frac{b}{\omega}\sin(\omega t)+a\cos(\omega t) \qquad \{9\}$$

Plugging in {6} and {7},

$$y_{ss}(t)=A\left[\text{Re}\left\{G(j\omega)\right\}\sin(\omega t)+\text{Im}\left\{G(j\omega)\right\}\cos(\omega t)\right] \qquad \{10\}$$

Or, using a trigonometric identity, we have:

$$y_{ss}(t)=A\cdot\left|G(j\omega)\right|\cdot\sin\left(\omega t+\angle G(j\omega)\right)\;;\;\angle G(j\omega)=\tan^{-1}\left(\frac{\text{Im}\left[G(j\omega)\right]}{\text{Re}\left[G(j\omega)\right]}\right) \qquad \{11\}$$