Basic Motor Model Analysis

January 1, 2015

Electrical equation:

$$V = Ri + L\frac{di}{dt} + k_v\dot{\theta} \tag{1}$$

Dynamics equation:

$$k_t i - T_{ext} - b\dot{\theta} = J\ddot{\theta} \tag{2}$$

At steady state, these become:

$$V = Ri + k_v\dot{\theta} \tag{3}$$

$$k_t i = T_{ext} + b\dot{\theta} \tag{4}$$

Solving eqn 4 for i and plugging into eqn 3:

$$\begin{aligned} V &= \frac{R}{k_t}\left(T_{ext} + b\dot{\theta}\right) + k_v\dot{\theta} \\ &= \frac{R}{k_t}T_{ext} + \left(\frac{Rb}{k_t} + k_v\right)\dot{\theta} \\ &= \frac{R}{k_t}T_{ext} + \frac{Rb + k_v k_t}{k_t}\dot{\theta} \end{aligned} \tag{5}$$

Solving for $\dot{\theta}$:

$$\dot{\theta}\left(V, T_{ext}\right) = \frac{k_t}{Rb + k_v k_t}V - \frac{R}{Rb + k_v k_t}T_{ext} \tag{6}$$

Solving eqn 4 for $\dot{\theta}$ and plugging into eqn 3:

$$\begin{aligned} V &= Ri + k_v\left(\frac{k_t}{b}i - \frac{1}{b}T_{ext}\right) \\ &= \left(R + \frac{k_v k_t}{b}\right)i - \frac{k_v}{b}T_{ext} \\ &= \frac{Rb + k_v k_t}{b}i - \frac{k_v}{b}T_{ext} \end{aligned} \tag{7}$$

Solving for i:

$$i\left(V, T_{ext}\right) = \frac{b}{Rb + k_v k_t}V + \frac{k_v}{Rb + k_v k_t}T_{ext} \tag{8}$$

Power in:

$$\begin{aligned} P_{in}\left(V, T_{ext}\right) &= Vi \\ &= \frac{b}{Rb + k_v k_t}V^2 + \frac{k_v}{Rb + k_v k_t}VT_{ext} \end{aligned} \tag{9}$$

Power out:

$$\begin{aligned} P_{out}\left(V, T_{ext}\right) &= T_{ext}\dot{\theta} \\ &= \frac{k_t}{Rb + k_v k_t}VT_{ext} - \frac{R}{Rb + k_v k_t}T_{ext}^2 \end{aligned} \tag{10}$$

Efficiency:

$$\begin{aligned}
\eta\left(V, T_{ext}\right) &= \frac{P_{out}}{P_{in}} \\
&= \frac{\frac{k_t}{Rb+k_v k_t} V T_{ext} - \frac{R}{Rb+k_v k_t} T_{ext}^2}{\frac{b}{Rb+k_v k_t} V^2 + \frac{k_v}{Rb+k_v k_t} V T_{ext}} \\
&= \frac{k_t V T_{ext} - R T_{ext}^2}{b V^2 + k_v V T_{ext}} \\
&= \frac{T_{ext}}{V}\left(\frac{k_t V - R T_{ext}}{b V + k_v T_{ext}}\right) \qquad (11)
\end{aligned}$$

Or, we can write the efficiency as a function of $\dot{\theta}$ and T_{ext}:

$$\begin{aligned}
\eta\left(\dot{\theta}, T_{ext}\right) &= \frac{T_{ext}\dot{\theta}}{\left(\frac{R}{k_t} T_{ext} + \frac{Rb+k_v k_t}{k_t}\dot{\theta}\right)\left(\frac{1}{k_t} T_{ext} + \frac{b}{k_t}\dot{\theta}\right)} \\
&= \frac{T_{ext}\dot{\theta}}{\frac{R}{k_t^2} T_{ext}^2 + \frac{b(Rb+k_v k_t)}{k_t^2}\dot{\theta}^2 + \left(\frac{2Rb+k_v k_t}{k_t^2}\right) T_{ext}\dot{\theta}} \qquad (12)
\end{aligned}$$

Power dissipated in motor:

$$\begin{aligned}
P_d\left(\dot{\theta}, T_{ext}\right) &= P_{in} - P_{out} \qquad (13) \\
&= \left(\frac{2Rb + k_v k_t}{k_t^2} - 1\right) T_{ext}\dot{\theta} + \frac{b\left(Rb + k_v k_t\right)}{k_t^2}\dot{\theta}^2 + \frac{R}{k_t^2} T_{ext}^2
\end{aligned}$$

Using the fact that $k_t = k_v$, this simplifies to:

$$P_d\left(\dot{\theta}, T_{ext}\right) = \frac{2Rb}{k_t^2} T_{ext}\dot{\theta} + \left(\frac{Rb}{k_t^2} + 1\right) b\dot{\theta}^2 + \frac{R}{k_t^2} T_{ext}^2$$