<u>Fundamental Therorem of Calculus</u>

11-1-05

There are many beautiful things about calculus, but one of the most amazing to me is the way differential calculus (the study of derivatives) and integral calculus (the study of summations) are related. (I know that this relation might seem second-nature to you, but that's because this relationship is one of the first things you learned and committed to memory in high-school calculus.)

Differential Calculus

Differential calculus is the study of rates. For functions of a single variable, the derivative tells us how the function value (or dependent variable) changes due to a change in the independent variable. Graphically, we interpret this as the "slope" of the function. The definition of the derivative of a function, f, of a single variable, x, is:

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \tag{1.1}$$

That's really all there is to differential calculus. Imagine some dude gave you a plot of the function f, and asked you to tell him the derivative (or slope) of f at a certain value of x. In other words, he wants to know the direction in which the function is pointing. You can do this graphically in about a second by simple eye-balling a straight line through the point of interest that is tangent (i.e. in the same direction) to the function at that point. Equation (1.1) is just a way of expressing what you've done in your head mathematically.

By the way, recall that when you take your first calculus course, you learn all of these cool tricks and methods for determining the derivatives of all different types of functions. For example, you learn that,

$$\frac{d}{dx}\left(ax^b\right) = abx^{b-1}.$$

There's a name for this "formula" that I can't remember, but that's not the point. It can easily be derived by just plugging the function in to the definition of the derivative (eqn 1.1). Let's try it for the function $f(x) = 3x^2$:

$$\begin{aligned}\frac{df}{dx} &= \lim_{\Delta x \to 0} \frac{3(x+\Delta x)^2 - 3(x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{3\left(x^2 + x\Delta x + \Delta x^2\right) - 3x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\left(3x^2 + 3x\Delta x + 3\Delta x^2\right) - 3x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x\Delta x + 3\Delta x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(3x + 3\Delta x\right) = 3x\end{aligned}$$

So there's nothing hard about all those weird derivative rules we learned in class. My main point is that from equation (1.1) alone, we are pretty much able master the art of derivative calculus. Things are not so easy for integral calculus…

Integral Calculus

Integral calculus is really just a study of addition. More specifically, it's the study of the following type of summation:

$$\lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x)\Delta x \tag{1.2}$$

The integral symbol was introduced just so we don't have to keep writing all that limit stuff all the time:

$$\int_a^b f(x)\,dx \equiv \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} f(x)\Delta x \tag{1.3}$$

But be careful with it. Don't loose track of the fact that an integral is really just a summation. At this point, it has absolutely nothing to do with derivatives or antiderivatives. For functions of a single variable, it's easy to think of the integral of a function as the area under the function. Figure 1 shows an example. We can approximate the area by using rectangles whose individual areas are equal to $f(x)\Delta x$. For big values of Δx, like those shown in the figure, the summation of all the individual rectangle areas will be different from the actual area under the curve, but as we shrink Δx towards zero, the summation approaches the actual area under the function. By the way, there's still nothing related to derivatives here!

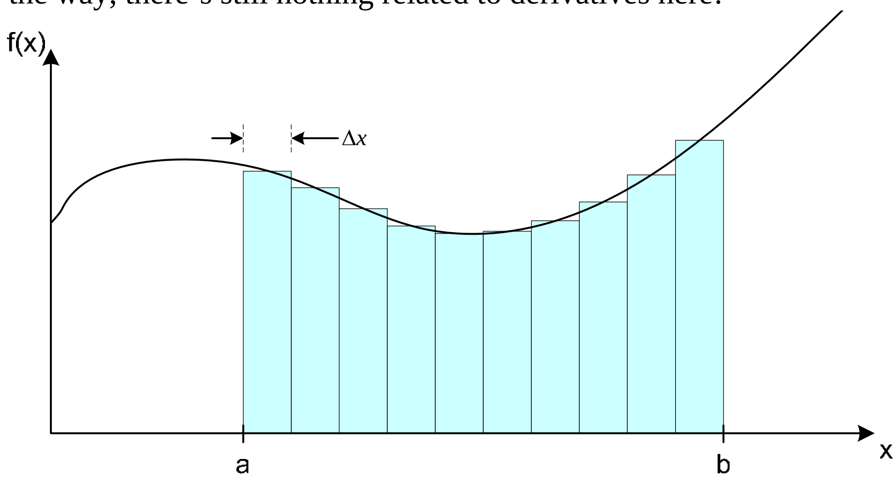

Figure 1. Graphical illustration of an integral

So now we've defined this thing called an integral, and it's pretty to easy to conceptualize what it represents. But there's one problem: Aside from some very simple types of functions, we can't really do anything with it! If the *f* is a straight line, then you're lucky, because you know how to calculate the area of a triangle (since the area under any straight line can be broken up into a rectangle and a triangle). For almost all other types of functions, we're stuck. You can't tell me the area under a parabola, for example, knowing what we know now.

The trick that will enable us to make use of the integral is called the Fundamental Theorem of Calculus.

Fundamental Theorem

The fundamental theorem of calculus relates differential calculus to integral calculus. Take a second to appreciate what I just said. Why should the two concepts of a derivative and an integral be related? They shouldn't! Derivatives and integrals were developed separately and for different reasons, but check this out…

Let's say we have a function, *f*, of a single variable, *x*. Suppose now that we write an integral expression that looks like,

$$\int_a^y f(x)\,dx. \tag{1.4}$$

Any funny business here? Well, the limits of integration are now from *a* to *y*. Why not from *a* to *b*? That would have been fine too, but I'm trying to set you up for something new. Now, if I give you the expression for *f*, and I give you values for *a* and *y*, can you tell me a value for the integral in (1.4)? Well, if *f* is a straight line, then yes, because we know how to calculate the area under a straight line. If *f* is something more complicated, then you won't be able to give me an exact answer, but you could give me a pretty close approximation by splitting up the interval between *a* and *y* into a zillion pieces, and using the old rectangle trick. The point is, no matter what *f* looks like, the integral in (1.4) is equal to some number. Does this number depend on the limits of integration? In other words, if I change the value for *a* and/or *y*, does the value of the integral change? Of course it does. From this point on, let's assume that *a* is constant, and cannot be changed. *y*, on the other hand, we are going to play with, and since we've concluded that the integral depends on the value of *y*, I'm going to express the integral in (1.4) as a function of *y*:

$$g(y) = \int_a^y f(x)\,dx \tag{1.5}$$

I promise there's nothing tricky here. I'm just giving our little integral a name (which is *g*), and saying that it depends on the value of *y*. So here we go. Let's try to write the derivative of g using the definition of the derivative:

$$\frac{dg}{dy} = \lim_{\Delta y \to 0} \frac{\int_a^{y+\Delta y} f(x)\,dx - \int_a^y f(x)\,dx}{\Delta y} \tag{1.6}$$

All I've done is literally plug our expression for *g* into equation (1.1). Now, we can use a property of addition (remember that an integral is just a summation!) and say that,

$$\int_a^{y+\Delta y} f(x)\,dx = \int_a^y f(x)\,dx + \int_y^{y+\Delta y} f(x)\,dx. \tag{1.7}$$

Plug this into (1.6), and you'll get,

$$\frac{dg}{dy} = \lim_{\Delta y \to 0} \frac{\int_a^y f(x)\,dx + \int_y^{y+\Delta y} f(x)\,dx - \int_a^y f(x)\,dx}{\Delta y} = \lim_{\Delta y \to 0} \frac{\int_y^{y+\Delta y} f(x)\,dx}{\Delta y} \tag{1.8}$$

The numerator of (1.8) is the integral of f between y and $y + \Delta y$. But this isn't any old Δy! This is the limit as Δy is being shrunk towards zero. So we can imagine the limits of integration in figure 1 being squeezed closer and closer to each other. Eventually, the area under the curve between these infinitely close limits of integration becomes just $f(y)\Delta y$, which is just the area of the infinitely thin rectangle with width Δy and height $f(y)$. (There are many ways to conceptualize this, and it is the only tricky thing about this whole derivation, so call me if it doesn't make sense.) So now, we can write the derivative of *g* as follows:

$$\frac{dg}{dy} = \lim_{\Delta y \to 0} \frac{f(y)\Delta y}{\Delta y} = f(y) \tag{1.9}$$

I'm now going to do something that used to make me uneasy. I'm going to make a change in variables in equation (1.9). Everywhere I see a *y*, I'm going to replace it with an *x*. The result is:

$$\frac{dg}{dx} = \lim_{\Delta x \to 0} \frac{f(x)\Delta x}{\Delta x} = f(x) \tag{1.10}$$

This used to weird me out, because I'd think "No way! *g* isn't a function of *x*, it's a function of *y*!" The important thing to remember is that *g* is a function of a single variable. It doesn't matter what you call that variable. You can replace *y* with any letter or symbol you want, and it still works.

Having done this, we're there! We showed that the derivative of *g* is equal to *f*. How does this help? Well, our main problem was that we could write down an integral expression (in this case *g*), we could conceptualize what it meant, we could even approximate it with the rectangle trick, but we couldn't write the actual value for it. Equation (1.10) tells us how to find *g* once we're given a function, *f*. For example, suppose we have $f(x) = 4x^2$, and we'd like to calculate *g*. Equation (1.10) tells us that the derivative of *g* has to be equal to *f* (which we already know). Our problem has become to find a function whose derivative is equal to $4x^2$. This is a job for differential calculus, which we've got totally figured out. We can just use one of our fancy derivative tricks (the same one mentioned above). The first answer that comes to mind is:

$$g(x) = \frac{4}{3}x^3 \quad \text{or} \quad g(y) = \frac{4}{3}y^3 \tag{1.11}$$

We can check this answer by simply taking its derivative, which is $4x^2$, as it should be. Now, I wish I could say we're totally done (to me we've already uncovered the coolest part), but we're not quite there yet. Your first clue might be the fact that our answer doesn't contain *a* in it. Recall that the definition of *g* is (written again as a function of *y*):

$$g(y) = \int_a^y f(x)\,dx. \tag{1.12}$$

We interpret this as the area under the function *f* between *x*=*a* and *x*=*y*. It's hard (if not impossible) to imagine how this area wouldn't depend on the value of *a*. Don't worry – it does. With some thought, we can realize that (1.11) does not represent the only possible expression for *g*. How about:

$$g(x) = \frac{4}{3}x^3 + 7.$$

Isn't its derivative equal to *f*? In fact there are an infinite number of expressions for *g* (all differing by just a constant) that satisfy equation (1.10). The general expression for g can be written:

$$g(x) = \frac{4}{3}x^3 + K \quad \text{or} \quad g(y) = \frac{4}{3}y^3 + K, \tag{1.13}$$

where *K* can be any number. So how then do we calculate the integral of $4x^2$ from $x=x_1$ to $x=x_2$? To do this, we use the same little addition trick we employed in (1.7):

$$\int_{x_1}^{x_2} f(x)\,dx = \int_{x_1}^{a} f(x)\,dx + \int_{a}^{x_2} f(x)\,dx = -\int_{a}^{x_1} f(x)\,dx + \int_{a}^{x_2} f(x)\,dx = -g(x_1) + g(x_2) \tag{1.14}$$

This is the second part of the fundamental theorem, and it tells us that:

$$\int_{x_1}^{x_2} f(x)\,dx = g(x_2) - g(x_1) \tag{1.15}$$

For our current example, we can write:

$$\int_{x_1}^{x_2} 4x^2 dx = \left(\frac{4}{3}x_2^3 + K\right) - \left(\frac{4}{3}x_1^3 + K\right) = \frac{4}{3}x_2^3 - \frac{4}{3}x_1^3 = \frac{4}{3}\left(x_2^3 - x_1^3\right) \tag{1.16}$$

Note how the constant, K, dropped out of the picture. In fact, it always will, because the integral of any function, *f*, will always be equated to the difference between two values of *g*, evaluated at the integration limits. This subtraction will always cancel out the integration constant, *K*.

We can substitute (1.10) into (1.15) to get the more common form of the fundamental theorem (the one you knew by heart before you started reading this whole thing):

$$\boxed{\int_{x_1}^{x_2} \frac{dg}{dx}\,dx = g(x_2) - g(x_1)} \tag{1.17}$$

This one formula relates the two fields of calculus that are seemingly unrelated.