**Derivation of the Complex Fourier Series from the sine/cosine Fourier Series**

Let the function *f(t)* be defined for $0 \le t \le T$. Let it also be periodic with period *T*. Then *f(t)* can be expressed in terms of an infinite series of sine and cosine terms, as follows:

$$\boxed{f(t) = \sum_{n=0}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)}, \tag{1.1}$$

where the Fourier coefficients $a_n$ and $b_n$ are given by,

$$\boxed{a_n = \begin{cases} \dfrac{1}{T}\displaystyle\int_0^T f(t)\,dt & ;\ n=0 \\ \dfrac{2}{T}\displaystyle\int_0^T f(t)\cos(n\omega_0 t)\,dt & ;\ n>0 \end{cases}} \tag{1.2}$$

$$\boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt} \tag{1.3}$$

Plug in exponential forms for sine and cosine:

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} \frac{a_n}{2}\left(e^{in\omega_0 t} + e^{-in\omega_0 t}\right) + \frac{b_n}{2i}\left(e^{in\omega_0 t} - e^{-in\omega_0 t}\right) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{2}\left(e^{in\omega_0 t} + e^{-in\omega_0 t}\right) - i\frac{b_n}{2}\left(e^{in\omega_0 t} - e^{-in\omega_0 t}\right) \\ &= \sum_{n=0}^{\infty} \left(\frac{a_n}{2} - i\frac{b_n}{2}\right)e^{in\omega_0 t} + \left(\frac{a_n}{2} + i\frac{b_n}{2}\right)e^{-in\omega_0 t} \\ &= \sum_{n=0}^{\infty} \left(\frac{a_n}{2} - i\frac{b_n}{2}\right)e^{in\omega_0 t} + \sum_{n=0}^{\infty}\left(\frac{a_n}{2} + i\frac{b_n}{2}\right)e^{-in\omega_0 t} \end{aligned} \tag{1.4}$$

Make the following substitution in the second summation: $n' = -n \quad \Rightarrow \quad n = -n'$

$$f(t) = \sum_{n=0}^{n=\infty} \left(\frac{a_n}{2} - i\frac{b_n}{2}\right)e^{in\omega_0 t} + \sum_{-n'=0}^{-n'=\infty}\left(\frac{a_{-n'}}{2} + i\frac{b_{-n'}}{2}\right)e^{in'\omega_0 t}$$

Now replace $n'$ with $n$:

$$f(t) = \sum_{n=0}^{\infty} \left(\frac{a_n}{2} - i\frac{b_n}{2}\right)e^{in\omega_0 t} + \sum_{n=0}^{-\infty}\left(\frac{a_{-n}}{2} + i\frac{b_{-n}}{2}\right)e^{in\omega_0 t}$$

This can be written as:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \tag{1.5}$$

$$c_n = \begin{cases} \dfrac{a_n}{2} - i\dfrac{b_n}{2} & ;\ n>0 \\ a_0 & ;\ n=0 \\ \dfrac{a_{-n}}{2} + i\dfrac{b_{-n}}{2} & ;\ n<0 \end{cases} = \begin{cases} \dfrac{a_n - ib_n}{2} & ;\ n>0 \\ a_0 & ;\ n=0 \\ \dfrac{a_{-n} + ib_{-n}}{2} & ;\ n<0 \end{cases} \tag{1.6}$$

Note that

$$c_{-k} = \overline{c_k} \qquad (1.7)$$

Now plug (1.2) and (1.3) into (1.6) for $n > 0$, which yields:

$$c_n = \frac{1}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt - i\frac{1}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt \quad ; \; n > 0$$

$$= \frac{1}{T}\int_0^T f(t)\left[\cos(n\omega_0 t) - i\sin(n\omega_0 t)\right]dt \quad ; \quad n > 0$$

Using Euler's relation, this reduces to:

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega_0 t}dt \;\; ; \text{ for } n > 0 \qquad (1.8)$$

Doing the same thing for $n < 0$, we get:

$$c_n = \frac{1}{T}\int_0^T f(t)\cos(-n\omega_0 t)\,dt + i\frac{1}{T}\int_0^T f(t)\sin(-n\omega_0 t)\,dt$$

$$= \frac{1}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt - i\frac{1}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt \quad ; \; n < 0$$

$$= \frac{1}{T}\int_0^T f(t)\left[\cos(n\omega_0 t) - i\sin(n\omega_0 t)\right]dt$$

Again using Euler's relation:

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega_0 t}dt \;\; ; \text{ for } n < 0 \qquad (1.9)$$

And finally, we look at the case of $n = 0$:

$$c_n = \frac{1}{T}\int_0^T f(t)\,dt \;\; ; \; n = 0$$

Which we realize is equal to:

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega_0 t}dt \;\; ; \; n = 0 \qquad (1.10)$$

So we've shown (exhaustively) that only one expression is needed for $c_n$, which applies for all values of *n*:

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega_0 t}dt \qquad (1.11)$$

Summarizing the Complex Fourier Series, we have:

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}} \qquad (1.12)$$

where,

$$\boxed{c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega_0 t}dt} \qquad (1.13)$$

**Continuous Fourier Transform for *f(t)* defined from *t=0* to *t*=∞ (Unilateral)**

Now let's consider the case where the period, *T*, of the function *f(t)* approaches infinity. First, let's define two new symbols:

$$\omega_n = n\omega_0 = n\frac{2\pi}{T} \tag{1.14}$$

and

$$\Delta\omega = \omega_{n+1} - \omega_n = \frac{2\pi}{T} \tag{1.15}$$

Using these, we can rewrite (1.12) and (1.13):

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \tag{1.16}$$

$$c_n = \frac{\Delta\omega}{2\pi}\int_0^T f(t)e^{-i\omega_n t}dt \tag{1.17}$$

And now plugging (1.17) into (1.16), we get,

$$f(t) = \sum_{n=-\infty}^{\infty}\left(\frac{\Delta\omega}{2\pi}\int_0^T f(t)e^{-i\omega_n t}dt\right)e^{i\omega_n t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_0^T f(t)e^{-i\omega_n t}dt\right)e^{i\omega_n t}\Delta\omega \tag{1.18}$$

Note from (1.14) that the discrete sequence of $\omega_n$'s approaches a continuous variable (call it $\omega$) as $T \to \infty$. Also note from (1.15) that $\Delta\omega$ approaches zero as $T \to \infty$.
We're now ready to apply the limiting case to *f(t)*:

$$\lim_{T\to\infty} f(t) = \lim_{T\to\infty}\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left(\int_0^T f(t)e^{-i\omega_n t}dt\right)e^{i\omega_n t}\Delta\omega \tag{1.19}$$

The integral in parenthesis becomes what we will call the Fourier Transform of *f(t)*:

$$\boxed{F(\omega) = \int_0^{+\infty} f(t)e^{-i\omega t}dt} \quad .$$

(1.20)

Plugging this into (1.19), we get

$$f(t) = \frac{1}{2\pi}\left(\lim_{T\to\infty}\sum_{n=-\infty}^{\infty} F(\omega)e^{i\omega t}\Delta\omega\right). \tag{1.21}$$

Now realize that the quantity in parenthesis is the definition of a Riemann integral:

$$\lim_{T\to\infty}\sum_{n=-\infty}^{\infty} F(\omega)e^{i\omega t}\Delta\omega = \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega,$$

which means,

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)e^{i\omega t}d\omega}. \tag{1.22}$$

**Continuous Fourier Transform for *f(t)* defined from *t*=-∞ to *t*=∞ (bilateral)**

Beginning with our original form of the complex Fourier coefficient:

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega_0 t}dt \qquad (1.23)$$

We can use the following change of variables to make the integral symmetric:

$$\tau = t - \frac{T}{2} \quad ; \quad t = \tau + \frac{T}{2} \quad ; \quad dt = d\tau$$

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f\left(\tau + \frac{T}{2}\right)e^{-in\omega_0\left(\tau+\frac{T}{2}\right)}d\tau$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} f\left(\tau + \frac{T}{2}\right)e^{-in\omega_0\frac{T}{2}}e^{-in\omega_0\tau}d\tau$$

$$= \frac{1}{T}(-1)^n\int_{-\frac{T}{2}}^{+\frac{T}{2}} f\left(\tau + \frac{T}{2}\right)e^{-in\omega_0\tau}d\tau$$

Now look at:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$$

$$f\left(\tau + \frac{T}{2}\right) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0\left(\tau+\frac{T}{2}\right)}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0\frac{T}{2}}e^{in\omega_0\tau}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{in\frac{2\pi}{T}\frac{T}{2}}e^{in\omega_0\tau}$$

$$= (-1)^n\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0\tau}$$

$$f\left(\tau+\frac{T}{2}\right)=(-1)^n\sum_{n=-\infty}^{\infty}\left[\frac{1}{T}(-1)^n\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(\tau+\frac{T}{2}\right)e^{-in\omega_0\tau}d\tau\right]e^{in\omega_0\tau}$$

$$=(-1)^n(-1)^n\sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(\tau+\frac{T}{2}\right)e^{-in\omega_0\tau}d\tau\right]e^{in\omega_0\tau} \qquad (1.24)$$

$$=\sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(\tau+\frac{T}{2}\right)e^{-in\omega_0\tau}d\tau\right]e^{in\omega_0\tau}$$

Let's introduce a new function g(t), such that:

$$g(\tau)=f\left(\tau+\frac{T}{2}\right) \qquad (1.25)$$

This means that the function *g* is equal to the function *f*, time-shifted to the left by half of its period. Plugging this into (1.24), we obtain,

$$g(\tau)=\sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(\tau)e^{-in\omega_0\tau}d\tau\right]e^{in\omega_0\tau}\ .$$

And after substituting $t=\tau$, we get:

$$g(t)=\sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)e^{-in\omega_0 t}dt\right]e^{in\omega_0 t}\ ,$$

which can be written as,

$$\boxed{g(t)=\sum_{n=-\infty}^{\infty} d_n e^{in\omega_0 t}} \qquad (1.26)$$

$$\boxed{d_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)e^{-in\omega_0 t}dt} \qquad (1.27)$$

Applying the limiting process of $T\rightarrow\infty$, the Fourier Transform for this symmetric case becomes:

$$\boxed{G(\omega)=\int_{-\infty}^{+\infty} g(t)e^{-i\omega t}dt} \qquad (1.28)$$

$$\boxed{g(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} G(\omega)e^{i\omega t}d\omega} \qquad (1.29)$$

## Discrete Fourier Series (Called the Discrete Fourier Transform)

Suppose I have a discrete-time data set, which represents a sampled version of the continuous function *f(t)*. The data points are equally spaced $T_s$ seconds apart, and there are a total of $(N+1)$ data points which span from $t_{start} = 0$ to $t_{end} = NT_s$. In this case, T is given by $NT_s$. In order to determine the frequency content in my data set, I need to calculate the complex Fourier coefficients given by (1.13). This means I need to approximate the integral in (1.13). Assuming my samples are sampled fast enough, I can approximate (1.13) by:

$$\begin{aligned}\hat{c}_n &= \frac{1}{T}\sum_{k=0}^{N} f(kT_s)e^{-in\omega_0(kT_s)}T_s \\ &= \frac{T_s}{T}\sum_{k=0}^{N} f(kT_s)e^{-in\omega_0(kT_s)} \\ &= \frac{1}{N}\sum_{k=0}^{N} f(kT_s)e^{-in\omega_0(kT_s)}\end{aligned} \tag{1.30}$$

Recalling that,

$$\omega_0 = \frac{2\pi}{T} \tag{1.31}$$

Plugging this in, (1.30) simplifies to:

$$\hat{c}_n = \frac{1}{N}\sum_{k=0}^{N} f(kT_s)e^{-i\frac{2\pi nk}{N}}. \tag{1.32}$$

### Laplace Transform

The (unilateral) Laplace transform of a continuous function, *f(t)*, is defined as,

$$L\{f(t)\} = F(s) \equiv \int_0^\infty f(t)e^{-st}dt, \tag{1.33}$$

where s is generally (but not always) assumed to be a complex parameter, often expressed as,

$$s = \sigma + j\omega. \tag{1.34}$$

### Z-Transform

The Z-transform is the discrete-time counterpart to the Laplace Transform.

## Appendix

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\begin{aligned}
I &= \int_{t_0}^{t_0+T} \sin^2\left(n\omega_0 t\right) dt \\
&= \int_{t_0}^{t_0+T} \sin^2\left(\frac{2\pi n}{T} t\right) dt \\
&= \int_{t_0}^{t_0+T} \left(\frac{e^{i\frac{2\pi n}{T}t} - e^{-i\frac{2\pi n}{T}t}}{2i}\right)^2 dt \\
&= -\frac{1}{4} \int_{t_0}^{t_0+T} \left(e^{i\frac{2\pi n}{T}t} - e^{-i\frac{2\pi n}{T}t}\right)^2 dt \\
&= -\frac{1}{4} \int_{t_0}^{t_0+T} \left(e^{i\frac{4\pi n}{T}t} + e^{-i\frac{4\pi n}{T}t} - 2e^{i\frac{2\pi n}{T}t} e^{-i\frac{2\pi n}{T}t}\right) dt \\
&= -\frac{1}{4} \int_{t_0}^{t_0+T} \left(e^{i\frac{4\pi n}{T}t} + e^{-i\frac{4\pi n}{T}t} - 2\right) dt \\
&= -\frac{1}{4} \int_{t_0}^{t_0+T} \left(2\cos\left(\frac{4\pi n}{T} t\right) - 2\right) dt \\
&= \int_{t_0}^{t_0+T} \frac{1}{2} - \frac{1}{2}\cos\left(\frac{4\pi n}{T} t\right) dt \\
&= \left[\frac{1}{2} t - \frac{T}{8\pi n} \sin\left(\frac{4\pi n}{T} t\right)\right]\Bigg|_{t_0}^{t_0+T} \\
&= \left[\frac{1}{2}\left(t_0 + T\right) - \frac{T}{8\pi n} \sin\left(\frac{4\pi n}{T}\left(t_0 + T\right)\right)\right] - \left[\frac{1}{2} t_0 - \frac{T}{8\pi n} \sin\left(\frac{4\pi n}{T} t_0\right)\right] \\
&= \frac{T}{2} - \frac{T}{8\pi n} \sin\left(\frac{4\pi n}{T}\left(t_0 + T\right)\right) + \frac{T}{8\pi n} \sin\left(\frac{4\pi n}{T} t_0\right) \\
&= \frac{T}{2} - \frac{T}{8\pi n}\left[\sin\left(\frac{4\pi n}{T}\left(t_0 + T\right)\right) - \sin\left(\frac{4\pi n}{T} t_0\right)\right] \\
&= \frac{T}{2} - \frac{T}{8\pi n}\left[\sin\left(\frac{4\pi n}{T} t_0\right)\cos\left(4\pi n\right) + \cos\left(\frac{4\pi n}{T} t_0\right)\sin\left(4\pi n\right) - \sin\left(\frac{4\pi n}{T} t_0\right)\right] \\
&= \frac{T}{2} - \frac{T}{8\pi n}\left[\sin\left(\frac{4\pi n}{T} t_0\right) - \sin\left(\frac{4\pi n}{T} t_0\right)\right] \\
&= \frac{T}{2}
\end{aligned}
$$