1 Velocity Field

The velocity field is given by:

$$\mathbf{v}(x,y,z,t) = u(x,y,z,t)\,\hat{\mathbf{i}} + v(x,y,z,t)\,\hat{\mathbf{j}} + w(x,y,z,t)\,\hat{\mathbf{k}} \tag{1}$$

The total time derivate of $u(x,y,z,t)$ is given by:

$$\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial t} \\ &= \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t} \\ &= \nabla u \cdot \mathbf{v} + \frac{\partial u}{\partial t} \end{aligned} \tag{2}$$

Similarly, we have:

$$\frac{dv}{dt} = \nabla v \cdot \mathbf{v} + \frac{\partial v}{\partial t} \tag{3}$$

$$\frac{dw}{dt} = \nabla w \cdot \mathbf{v} + \frac{\partial w}{\partial t} \tag{4}$$

In vector form, we'll write it as:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \nabla \mathbf{v} \cdot \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \tag{5}$$

2 Conservation of Mass

The total rate of mass flow into any closed region, R, must equal the rate of change of mass contained in R:

$$-\iint\limits_R \rho\mathbf{v} \cdot d\mathbf{A} = \frac{\partial}{\partial t}\iiint\limits_R \rho dV \tag{6}$$

The left side surface integral can be converted to a volume integral using Gauss' Divergence theorem:

$$\iint\limits_R \rho\mathbf{v} \cdot d\mathbf{A} = \iiint\limits_R \nabla \cdot (\rho\mathbf{v})\, dV \tag{7}$$

Plugging this in, we get:

$$\iiint\limits_R \left[\nabla \cdot (\rho\mathbf{v}) + \frac{\partial \rho}{\partial t}\right] dV = 0 \tag{8}$$

Equating the integrand to zero, we get the differential form of conservation of mass:

$$\nabla \cdot (\rho\mathbf{v}) + \frac{\partial \rho}{\partial t} = 0 \tag{9}$$

For the case of incompressible flow (ie. constant density), this reduces to:

$$\nabla \cdot \mathbf{v} = 0 \tag{10}$$

3 Newton's Second Law (inviscid flow)

Consider an infinitesimal fluid element with dimensions dx, dy and dz. Assume the only forces acting on the element are due to pressure (normal stresses) and gravity. Writing Newton's Second law in the x-direction:

$$(p_x - p_{x+dx})\, dydz = \rho dxdydz \frac{du}{dt} \tag{11}$$

Recognizing $(p_x - p_{x+dx})$ as $\frac{\partial p}{\partial x}dx$, and cancelling $dxdydz$ from both sides, this implifies to:

$$\begin{aligned} -\frac{\partial p}{\partial x} &= \rho \frac{du}{dt} \\ &= \rho \left(\nabla u \cdot \mathbf{v} + \frac{\partial u}{\partial t} \right) \end{aligned} \tag{12}$$

Rearranging slightly:

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \nabla u \cdot \mathbf{v} + \frac{\partial u}{\partial t} \tag{13}$$

In the y-direction, we have the extra force of gravity:

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} - g = \nabla v \cdot \mathbf{v} + \frac{\partial v}{\partial t} \tag{14}$$

We can write Newton's 2nd law in vector form as follows:

$$-\frac{1}{\rho}\nabla p - g\hat{\mathbf{j}} = \nabla \mathbf{v} \cdot \mathbf{v} + \frac{\partial \mathbf{v}}{\partial t} \tag{15}$$