A first (baby) step at understanding how to accurately predict closed-loop stability under digital control without making the assumption of a sufficiently high sampling rate

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Consider the closed-loop control system in the figure below. The continuous plant has a simple first order transfer function, and the digital controller is of the proportional type. The signal output from the D/A is assumed to be zero-order held, as is typically the case.

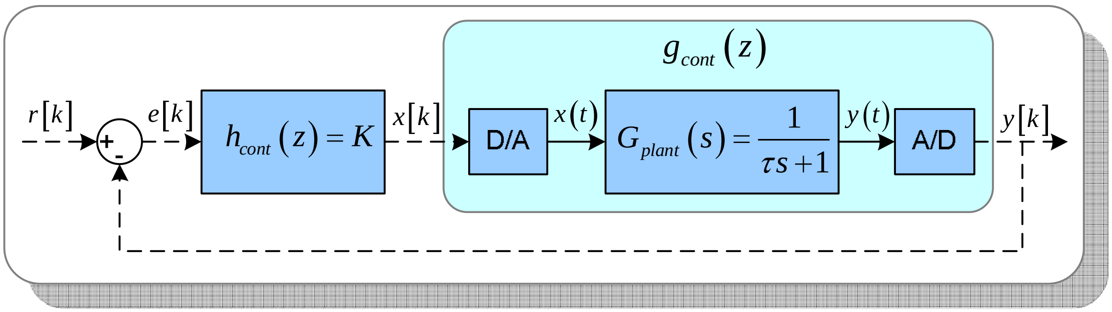

Figure 1. Simple closed loop control system with discrete-time controller

Assume the plant TF is given by:

$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{1}{\tau s+1} \tag{1}$$

This corresponds to the following differential equation in the time-domain:

$$\tau \dot{y} + y = x \tag{2}$$

Manipulating this, we get:

$$\dot{y} + \frac{1}{\tau} y = \frac{1}{\tau} x \tag{3}$$

Multiplying by $e^{\frac{t}{\tau}}$, we get:

$$e^{\frac{t}{\tau}}\left(\dot{y} + \frac{1}{\tau} y\right) = \frac{d}{dt}\left(ye^{\frac{t}{\tau}}\right) = e^{\frac{t}{\tau}}\frac{1}{\tau} x \tag{4}$$

Integrating both sides from t_0 to $t_0 + T_s$, where t_0 corresponds to a sampling strobe:

$$\begin{aligned} &\int_{t_0}^{t_0+T_s} \frac{d}{dt}\left(ye^{\frac{t}{\tau}}\right) dt = \left(ye^{\frac{t}{\tau}}\right)\Bigg|_{t_0}^{t_0+T_s} = \int_{t_0}^{t_0+T_s} e^{\frac{t}{\tau}} \frac{1}{\tau} x(t)\, dt \\ &\Rightarrow y(t_0+T_s)e^{\frac{t_0+T_s}{\tau}} - y(t_0)e^{\frac{t_0}{\tau}} = \frac{1}{\tau}\int_{t_0}^{t_0+T_s} e^{\frac{t}{\tau}} x(t)\, dt \end{aligned} \tag{5}$$

Noting that $x(t)$ is constant throughout the interval of integration (thanks to the ZOH):

$$
\begin{aligned}
y(t_0+T_s)e^{\frac{t_0+T_s}{\tau}} &= y(t_0)e^{\frac{t_0}{\tau}} + \frac{1}{\tau}x(t_0)\int_{t_0}^{t_0+T_s} e^{\frac{t}{\tau}}dt \\
&= y(t_0)e^{\frac{t_0}{\tau}} + \frac{1}{\tau}x(t_0)\left[\tau e^{\frac{t}{\tau}}\right]_{t_0}^{t_0+T_s} \\
&= y(t_0)e^{\frac{t_0}{\tau}} + x(t_0)\left(e^{\frac{t_0+T_s}{\tau}} - e^{\frac{t_0}{\tau}}\right)
\end{aligned} \tag{6}
$$

Solving for $y(t_0+T_s)$

$$
\begin{aligned}
y(t_0+T_s) &= y(t_0)e^{-\frac{t_0+T_s}{\tau}}e^{\frac{t_0}{\tau}} + x(t_0)e^{-\frac{t_0+T_s}{\tau}}\left(e^{\frac{t_0+T_s}{\tau}} - e^{\frac{t_0}{\tau}}\right) \\
&= y(t_0)e^{-\frac{T_s}{\tau}} + x(t_0)\left(1-e^{-\frac{T_s}{\tau}}\right)
\end{aligned} \tag{7}
$$

Since t_0 corresponds to a sampling strobe, we can write:

$$t_0 = k\cdot T_s \tag{8}$$

and

$$t_0 + T_s = (k+1)\cdot T_s \tag{9}$$

where k is an integer. Introducing the following notation for a discrete time sequence:

$$y[n] = y(n\cdot T_s)\ , \tag{10}$$

we can rewrite (7) as follows:

$$y[k+1] = \underbrace{e^{-\frac{T_s}{\tau}}}_{\alpha}\cdot y[k] + \underbrace{\left(1-e^{-\frac{T_s}{\tau}}\right)}_{\beta}\cdot x[k]\ , \tag{11}$$

where we note that α and β are both constants for a given sample period and plant configuration. Taking the z-transform of both side and rearranging:

$$(z-\alpha)y(z) = \beta x(z) \Rightarrow \frac{y(z)}{x(z)} = \frac{\beta}{z-\alpha} \tag{12}$$

Plugging back in for α and β:

$$g_{plant}(z) = \frac{y(z)}{x(z)} = \frac{1-e^{-\frac{T_s}{\tau}}}{z-e^{-\frac{T_s}{\tau}}} \tag{13}$$

This describes the exact discrete time transfer function corresponding to the continuous plant being driven by a zero-order held signal.

Let’s assume a simple proportional controller:

$$g_{cont}(z) = k \tag{14}$$

The closed-loop z-domain TF is given by:

$$
\begin{aligned}
g_{CL}(z) &= \frac{g_{cont}(z)\,g_{plant}(z)}{1+g_{cont}(z)\,g_{plant}(z)} \\
&= \frac{k\dfrac{1-e^{-\frac{T_s}{\tau}}}{z-e^{-\frac{T_s}{\tau}}}}{1+k\dfrac{1-e^{-\frac{T_s}{\tau}}}{z-e^{-\frac{T_s}{\tau}}}} = \frac{k\left(1-e^{-\frac{T_s}{\tau}}\right)}{z-e^{-\frac{T_s}{\tau}}+k\left(1-e^{-\frac{T_s}{\tau}}\right)} \\
&= \frac{k\left(1-e^{-\frac{T_s}{\tau}}\right)}{z-\underbrace{\left[e^{-\frac{T_s}{\tau}}-k\left(1-e^{-\frac{T_s}{\tau}}\right)\right]}_{p_1}}
\end{aligned}
\tag{15}
$$

where the bracketed term represents the closed loop pole, p_1. The criterion for marginal stability for this system is the magnitude of the pole must be unity:

$$
\left| e^{-\frac{T_s}{\tau}} - k_m\left(1-e^{-\frac{T_s}{\tau}}\right)\right| = 1 \tag{16}
$$

This can happen two possible ways. The first is:

$$
\begin{aligned}
& e^{-\frac{T_s}{\tau}} - k_m\left(1-e^{-\frac{T_s}{\tau}}\right) = 1 \;\Rightarrow\; k_m\left(1-e^{-\frac{T_s}{\tau}}\right) = e^{-\frac{T_s}{\tau}} - 1 \\
& \Rightarrow\; k_m = -1
\end{aligned}
\tag{17}
$$

This solution makes no sense, because our controller can only provide position gain. The second possibility is:

$$
\begin{aligned}
& e^{-\frac{T_s}{\tau}} - k_m\left(1-e^{-\frac{T_s}{\tau}}\right) = -1 \;\Rightarrow\; k_m\left(1-e^{-\frac{T_s}{\tau}}\right) = 1 + e^{-\frac{T_s}{\tau}} \\
& \Rightarrow\; k_m = \frac{1+e^{-\frac{T_s}{\tau}}}{1-e^{-\frac{T_s}{\tau}}}
\end{aligned}
\tag{18}
$$

This solution is the one we're looking for. It determines the critical controller gain that corresponds to the system being marginally stable. Rewriting this result in terms of the sampling rate, f_s, and the plant's cut-off frequency, $f_c = \dfrac{1}{2\pi\tau}$:

$$
\boxed{k_m = \frac{1+e^{-\frac{2\pi}{f_s/f_c}}}{1-e^{-\frac{2\pi}{f_s/f_c}}}} \tag{19}
$$

One interesting thing to note is what happens to k_m as the ratio f_s / f_c approaches infinity. This corresponds to sampling at a "sufficiently" high rate.

$$\lim_{f_s/f_c \to \infty} k_m = \frac{1+\left(1-\frac{2\pi}{f_s/f_c}\right)}{1-\left(1-\frac{2\pi}{f_s/f_c}\right)} = \frac{2-\frac{2\pi}{f_s/f_c}}{\frac{2\pi}{f_s/f_c}} = \frac{f_s/f_c}{\pi} \tag{20}$$

This surprisingly simple formula determines the critical gain, k_m, for sufficiently high sampling rates.

At the other extreme, we can examine what happens to the critical gain as the sampling rate gets really low (i.e. as the ratio f_s / f_c approaches zero). From (19), we see that:

$$\lim_{f_s/f_c \to 0} k_m = 1 \tag{21}$$

Fixed Time delay approach (for sufficiently high sampling rates)

If we try to model the system in the continuous domain, and assume that the finite sampling period, T_s, results in effectively a fixed time delay (of $T_d = a \cdot T_s$), the magnitude and phase of the open loop TF are given by:

$$|G_{OL}| = \frac{k}{\sqrt{\tau^2\omega^2+1}} \tag{22}$$

and

$$\angle G_{OL} = -\tan^{-1}(\omega\tau) - \omega T_d \tag{23}$$

For relatively large values of $\tau\omega$, these become:

$$|G_{OL}| = \frac{k}{\tau\omega} \tag{24}$$

and

$$\angle G_{OL} = -\frac{\pi}{2} - \omega T_d \tag{25}$$

At the point of marginal stability, we know that $|G_{OL}| = 1$ while $\angle G_{OL} = -\pi$ (at the same frequency). Using this information, we can use (24) to solve for this cross-over frequency:

$$\omega_x = \frac{k_m}{\tau} \tag{26}$$

Plugging this into (25) for the marginal stability case, we can solve for the critical gain as a function of the fixed time lag.

$$\begin{aligned} -\pi &= -\frac{\pi}{2} - \omega_x T_d = -\frac{\pi}{2} - \frac{k_m}{\tau} T_d \\ \Rightarrow &\ \frac{k_m}{\tau} T_d = \frac{\pi}{2} \\ \Rightarrow &\ k_m = \frac{\pi\tau}{2T_d} = \frac{\pi\tau}{2aT_s} \end{aligned} \tag{27}$$

Rewriting this in terms of f_s / f_c, we get:

$$k_m = \frac{f_s / f_c}{4a} \tag{28}$$

Equating (28) to (20), we can determine the value of *a* that makes this approach yield the same stability results as the exact method described above (that is, for relatively large values of f_s / f_c):

$$\frac{f_s / f_c}{\pi} = \frac{f_s / f_c}{4a} \Rightarrow a = \frac{\pi}{4} \approx 0.785 \tag{29}$$

In other words, assuming the sampling rate is sufficiently high, the stability of the first order closed loop system in figure 1 can accurately be predicted by modeling the discrete-time effects of the digital controller as a fixed time delay equal to $T_d = 0.785 \cdot T_s$.

Fixed Time delay approach (for any sampling rate)

If we try to model the system in the continuous domain, and assume that the finite sampling period, T_s, results in effectively a fixed time delay (of $T_d = a \cdot T_s$), the magnitude and phase of the open loop TF are given by:

$$|G_{OL}| = \frac{k}{\sqrt{\tau^2 \omega^2 + 1}} \tag{30}$$

and

$$\angle G_{OL} = -\tan^{-1}(\omega \tau) - \omega T_d \tag{31}$$

At the point of marginal stability, we know that $|G_{OL}| = 1$ while $\angle G_{OL} = -\pi$ (at the same frequency). Therefore we can write:

$$\frac{k_m}{\sqrt{\tau^2 \omega_x^2 + 1}} = 1 \tag{32}$$

and

$$\pi = \tan^{-1}(\omega_x \tau) + \omega_x a T_s \tag{33}$$

We cannot solve this pair of equations in closed form fro ω_x and a, but we can solve them numerically. The plots in figures 2 and 3 compare calculated values of k_m using the fixed time delay approach to the exact solution over a range of f_s / f_c. Curves corresponding to three different time delays are plotted: $T_d = T_s / 2$ in red, $T_d = 0.785 \cdot T_s$ in blue (as derived above), and $T_d = T_s$ in green. The red and green traces are included only because they are commonly mentioned when the subject of discrete time controller effects is discussed. Note how the blue trace agrees very well with the exact solution for $f_s / f_c \geq 10$. However, for lower frequency ratios, the two curves diverge.

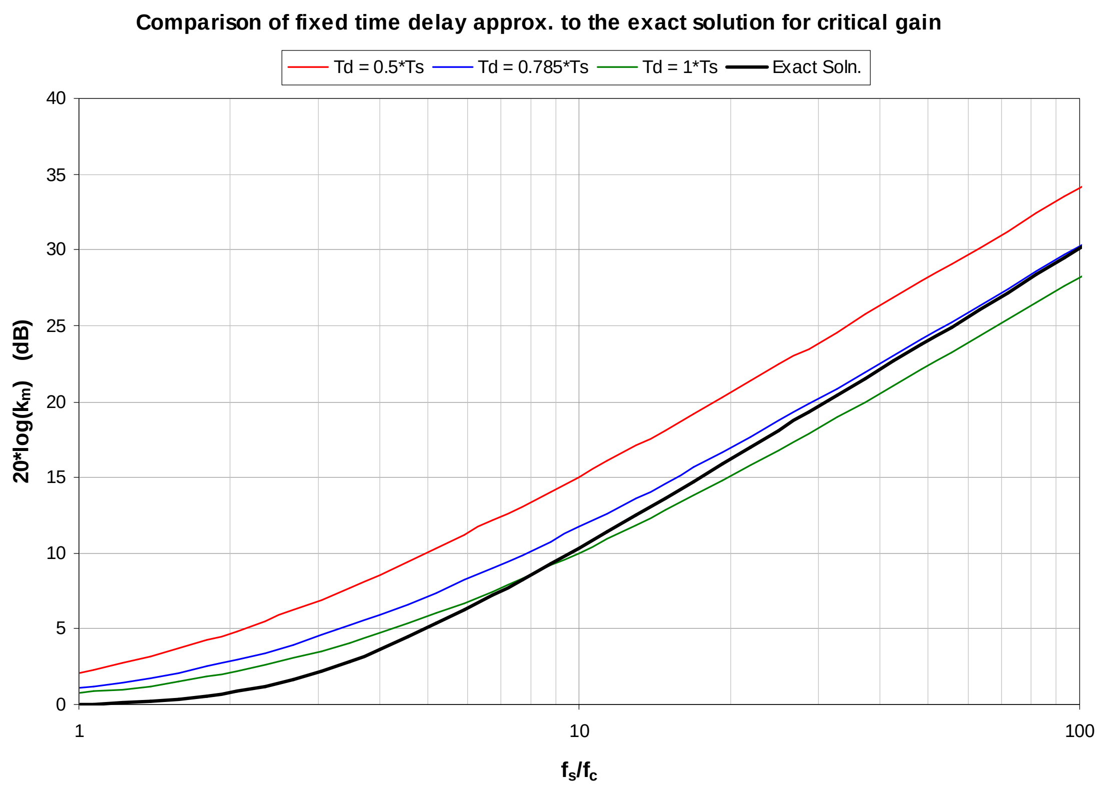

Figure 2.

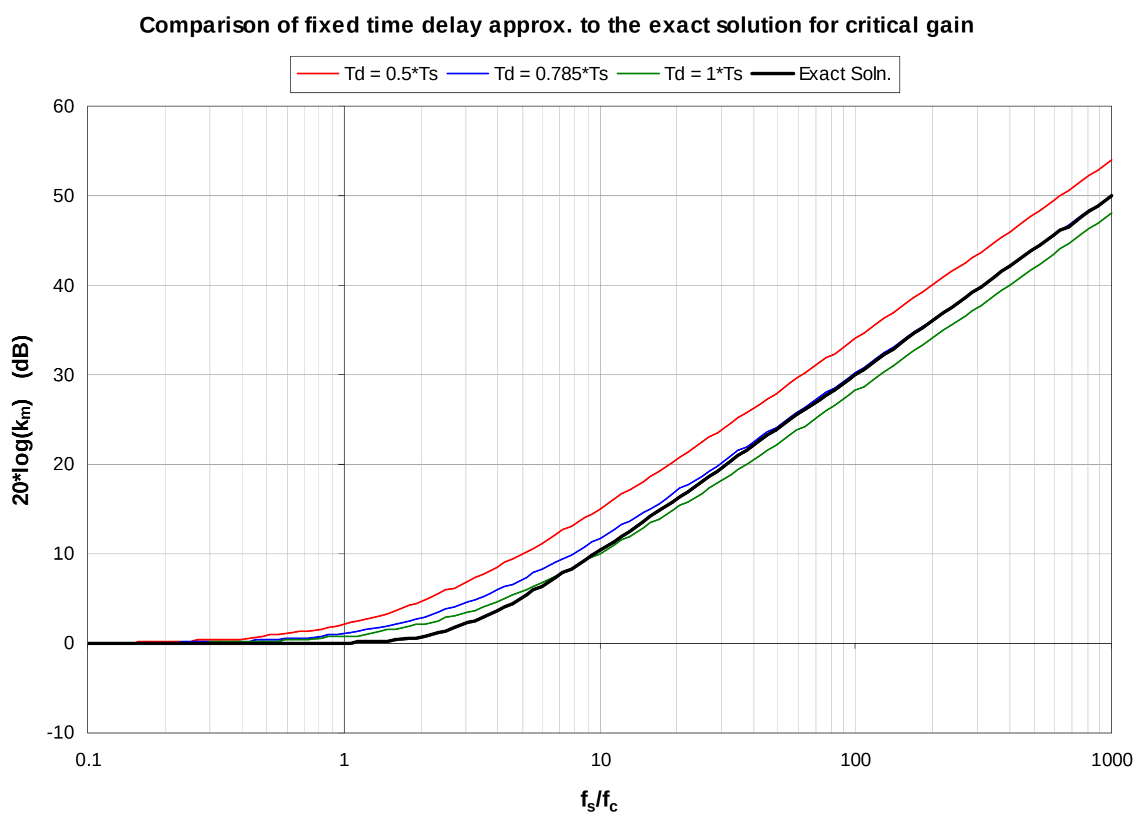

Figure 3