

Digital Filter Design Basics

Suppose we have a digital filter with k coefficients, whose output at time step n depends upon the input at time steps $(n-k)$ thru n .

$$y_n = a_0 x_n + a_1 x_{n-1} + a_2 x_{n-2} + \dots + a_k x_{n-k} = \sum_{i=0}^k a_i x_{n-i} \quad (1.1)$$

Assume an input of the form:

$$x(t) = e^{j\omega t} \Rightarrow x_n = e^{j\omega T_s n}$$

where T_s is the sample period. Plugging into (1.1), we get,

$$\begin{aligned} y_n &= a_0 e^{j\omega T_s n} + a_1 e^{j\omega T_s (n-1)} + a_2 e^{j\omega T_s (n-2)} + \dots + a_k e^{j\omega T_s (n-k)} \\ &= \underbrace{e^{j\omega T_s n}}_{x_n} (a_0 + a_1 e^{-j\omega T_s} + a_2 e^{-j2\omega T_s} + \dots + a_k e^{-jk\omega T_s}) \end{aligned}$$

Solving for the ratio output to input:

$$\begin{aligned} \frac{y_n}{x_n} &= a_0 + a_1 e^{-j\omega T_s} + a_2 e^{-j2\omega T_s} + \dots + a_k e^{-jk\omega T_s} \\ &= a_0 + a_1 [\cos(\omega T_s) - j \sin(\omega T_s)] + a_2 [\cos(2\omega T_s) - j \sin(2\omega T_s)] + \dots + a_k [\cos(k\omega T_s) - j \sin(k\omega T_s)] \\ &= [a_0 + a_1 \cos(\omega T_s) + a_2 \cos(2\omega T_s) + \dots + a_k \cos(k\omega T_s)] - j [a_1 \sin(\omega T_s) + a_2 \sin(2\omega T_s) + \dots + a_k \sin(k\omega T_s)] \end{aligned}$$

The magnitude of this ratio is:

$$\begin{aligned} \left| \frac{y_n}{x_n} \right| &= \sqrt{ \left[a_0 + a_1 \cos(\omega T_s) + a_2 \cos(2\omega T_s) + \dots + a_k \cos(k\omega T_s) \right]^2 + \left[a_1 \sin(\omega T_s) + a_2 \sin(2\omega T_s) + \dots + a_k \sin(k\omega T_s) \right]^2 } \\ &= \sqrt{ \left[\begin{aligned} &a_0^2 + a_1^2 \cos^2(\omega T_s) + a_2^2 \cos^2(2\omega T_s) + \dots + a_k^2 \cos^2(k\omega T_s) + \\ &2a_0 a_1 \cos(\omega T_s) + 2a_0 a_2 \cos(2\omega T_s) + \dots + 2a_0 a_k \cos(k\omega T_s) + \\ &2a_1 a_2 \cos(\omega T_s) \cos(2\omega T_s) + \dots + 2a_1 a_k \cos(\omega T_s) \cos(k\omega T_s) + \dots \\ &2a_2 a_3 \cos(2\omega T_s) \cos(3\omega T_s) + \dots + 2a_2 a_k \cos(2\omega T_s) \cos(k\omega T_s) + \dots \end{aligned} \right] + \left[\begin{aligned} &a_1^2 \sin^2(\omega T_s) + a_2^2 \sin^2(2\omega T_s) + \dots + a_k^2 \sin^2(k\omega T_s) + \\ &2a_1 a_2 \sin(\omega T_s) \sin(2\omega T_s) + \dots + 2a_1 a_k \sin(\omega T_s) \sin(k\omega T_s) + \dots \\ &2a_2 a_3 \sin(2\omega T_s) \sin(3\omega T_s) + \dots + 2a_2 a_k \sin(2\omega T_s) \sin(k\omega T_s) + \dots \end{aligned} \right] } \end{aligned}$$

Using the trig identities:

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\cos^2(x) + \sin^2(x) = 1$$

this simplifies to,

$$\begin{aligned}
\left| \frac{y_n}{x_n} \right| &= \text{sqrt} \left\{ \begin{aligned} &a_0^2 + a_1^2 + a_2^2 + \dots + a_k^2 + \\ &2a_0 [a_1 \cos(\omega T_s) + a_2 \cos(2\omega T_s) + \dots + a_k \cos(k\omega T_s)] + \\ &2a_1 [a_2 \cos(\omega T_s) + a_3 \cos(2\omega T_s) + \dots + a_k \cos((k-1)\omega T_s)] + \\ &2a_2 [a_3 \cos(\omega T_s) + a_4 \cos(2\omega T_s) + \dots + a_k \cos((k-2)\omega T_s)] + \dots \\ &+ 2a_{k-2} [a_{k-1} \cos(\omega T_s) + a_k \cos(2\omega T_s)] + 2a_{k-1} a_k \cos(\omega T_s) \end{aligned} \right\} \\
&= \text{sqrt} \left\{ \begin{aligned} &a_0^2 + a_1^2 + a_2^2 + \dots + a_k^2 + \\ &2(a_0 a_1 + a_1 a_2 + a_2 a_3 + \dots + a_{k-1} a_k) \cos(\omega T_s) + \\ &2(a_0 a_2 + a_1 a_3 + a_2 a_4 + \dots + a_{k-2} a_k) \cos(2\omega T_s) + \\ &2(a_0 a_3 + a_1 a_4 + a_2 a_5 + \dots + a_{k-3} a_k) \cos(3\omega T_s) + \dots \\ &2(a_0 a_{k-1} + a_1 a_k) \cos((k-1)\omega T_s) + 2a_0 a_k \cos((k-1)\omega T_s) \end{aligned} \right\} \\
&= \text{sqrt} \left\{ \begin{aligned} &\sum_{i=0}^k a_i^2 + 2 \cos(\omega T_s) \sum_{i=0}^{k-1} a_i a_{i+1} + 2 \cos(2\omega T_s) \sum_{i=0}^{k-2} a_i a_{i+2} + \\ &2 \cos(3\omega T_s) \sum_{i=0}^{k-3} a_i a_{i+3} + \dots + 2 \cos((k-1)\omega T_s) \sum_{i=0}^1 a_i a_{i+k-1} + \\ &2 \cos(k\omega T_s) a_0 a_k \end{aligned} \right\}
\end{aligned}$$

Finally, this can be written as a double summation:

$$\boxed{\left| G(\omega) \right| = \left| \frac{y_n}{x_n} \right| = \sqrt{\sum_{i=0}^k a_i^2 + \sum_{j=1}^k \sum_{i=0}^{k-j} 2a_i a_{i+j} \cos(j\omega T_s)}} \quad (1.2)$$

This can be implemented in MATLAB with the following script:

```

C:\matlabR12\work\df.m
File Edit View Text Debug Breakpoints Web Window Help
1  % This plots the Magnitude of the the TF for a
2  % digital filter of the form:
3  % y[n]=a0*x[n]+a1*x[n-1]+a2*x[n-2]+...+a_k*x[n-k]
4
5  a=[1 1 1 1]; % filter coefficients
6
7  Ts=1; % sample period
8  w_s=2*pi/Ts; % sample freq (rads/sec)
9  w_ny=w_s/2; % Nyquist freq (rads/sec)
10
11 k=length(a)-1;
12
13 w=linspace(0,w_s,200); % make frequency array
14
15 summer=0;
16 for j=1:1:k
17     for i=0:1:k-j
18         summer=summer+2*a(i+1)*a(i+1+j)*cos(j*Ts*w);
19     end
20 end
21
22 summer1=0;
23 for i=0:1:k
24     summer1=summer1+(a(i+1)^2);
25 end
26
27 Mag=sqrt(summer1+summer);
28
29 plot(w,Mag);
Ready

```

Example

Consider the simple derivative filter given by:

$$y_n = \frac{x_n - x_{n-1}}{T_s}$$

for which,

$$a_0 = \frac{1}{T_s}$$

and

$$a_1 = -\frac{1}{T_s}$$

Plugging into (1.2):

$$\begin{aligned}
|G(\omega)| &= \sqrt{\sum_{i=0}^1 a_i^2 + \sum_{j=1}^1 \sum_{i=0}^{1-j} 2a_i a_{i+j} \cos(j\omega T_s)} \\
&= \sqrt{\sum_{i=0}^1 a_i^2 + \sum_{i=0}^0 2a_i a_{i+1} \cos(\omega T_s)} \\
&= \sqrt{a_0^2 + a_1^2 + 2a_0 a_1 \cos(\omega T_s)} \\
&= \sqrt{\frac{2}{T_s^2} - \frac{2}{T_s^2} \cos(\omega T_s)} \\
&= \frac{1}{T_s} \sqrt{2 - 2 \cos(\omega T_s)}
\end{aligned}$$

which is the correct result. The plot of the TF is shown below:

