Digital Filter Design Basics

Suppose we have a digital filter with *k* coefficients, whose output at time step *n* depends upon the input at time steps *(n-k)* thru *n*.

$$y_n = a_0 x_n + a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k x_{n-k} = \sum_{i=0}^{k} a_i x_{n-i} \tag{1.1}$$

Assume an input of the form:

$$x(t) = e^{j\omega t} \;\Rightarrow\; x_n = e^{j\omega T_s n}$$

where T_s is the sample period. Plugging into (1.1), we get,

$$\begin{aligned} y_n &= a_0 e^{j\omega T_s n} + a_1 e^{j\omega T_s (n-1)} + a_2 e^{j\omega T_s (n-2)} + \cdots + a_k e^{j\omega T_s (n-k)} \\ &= \underbrace{e^{j\omega T_s n}}_{x_n} \left(a_0 + a_1 e^{-j\omega T_s} + a_2 e^{-j2\omega T_s} + \cdots + a_k e^{-jk\omega T_s} \right) \end{aligned}$$

Solving for the ratio output to input:

$$\begin{aligned} \frac{y_n}{x_n} &= a_0 + a_1 e^{-j\omega T_s} + a_2 e^{-j2\omega T_s} + \cdots + a_k e^{-jk\omega T_s} \\ &= a_0 + a_1 \left[\cos(\omega T_s) - j\cos(\omega T_s) \right] + a_2 \left[\cos(2\omega T_s) - j\cos(2\omega T_s) \right] + \cdots + a_k \left[\cos(k\omega T_s) - j\cos(k\omega T_s) \right] \\ &= \left[a_0 + a_1 \cos(\omega T_s) + a_2 \cos(2\omega T_s) + \cdots a_k \cos(k\omega T_s) \right] - j\left[a_1 \sin(\omega T_s) + a_2 \sin(2\omega T_s) + \cdots a_k \sin(k\omega T_s) \right] \end{aligned}$$

The magnitude of this ratio is:

$$\left| \frac{y_n}{x_n} \right| = sqrt \left\{ \begin{matrix} \left[a_0 + a_1 \cos(\omega T_s) + a_2 \cos(2\omega T_s) + \cdots a_k \cos(k\omega T_s) \right]^2 + \\ \left[a_1 \sin(\omega T_s) + a_2 \sin(2\omega T_s) + \cdots a_k \sin(k\omega T_s) \right]^2 \end{matrix} \right\}$$

$$= sqrt \left\{ \begin{matrix} \left[\begin{matrix} a_0^2 + a_1^2 \cos^2(\omega T_s) + a_2^2 \cos^2(2\omega T_s) + \cdots + a_k^2 \cos^2(k\omega T_s) + \\ 2a_0 a_1 \cos(\omega T_s) + 2a_0 a_2 \cos(2\omega T_s) + \cdots + 2a_0 a_k \cos(k\omega T_s) + \\ 2a_1 a_2 \cos(\omega T_s)\cos(2\omega T_s) + \cdots + 2a_1 a_k \cos(\omega T_s)\cos(k\omega T_s) + \cdots \\ 2a_2 a_3 \cos(2\omega T_s)\cos(3\omega T_s) + \cdots + 2a_2 a_k \cos(2\omega T_s)\cos(k\omega T_s) + \cdots \end{matrix} \right] + \\ \left[\begin{matrix} a_1^2 \sin^2(\omega T_s) + a_2^2 \sin^2(2\omega T_s) + \cdots + a_k^2 \sin^2(k\omega T_s) + \\ 2a_1 a_2 \sin(\omega T_s)\sin(2\omega T_s) + \cdots + 2a_1 a_k \sin(\omega T_s)\sin(k\omega T_s) + \cdots \\ 2a_2 a_3 \sin(2\omega T_s)\sin(3\omega T_s) + \cdots + 2a_2 a_k \sin(2\omega T_s)\sin(k\omega T_s) + \cdots \end{matrix} \right] \end{matrix} \right\}$$

Using the trig identities:

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

$$\cos^2(x) + \sin^2(x) = 1$$

this simplifies to,

$$\left|\frac{y_n}{x_n}\right| = sqrt\left\{\begin{array}{l} a_0^2 + a_1^2 + a_2^2 + \cdots + a_k^2 + \\ 2a_0\left[a_1\cos(\omega T_s) + a_2\cos(2\omega T_s) + \cdots + a_k\cos(k\omega T_s)\right] + \\ 2a_1\left[a_2\cos(\omega T_s) + a_3\cos(2\omega T_s) + \cdots + a_k\cos((k-1)\omega T_s)\right] + \\ 2a_2\left[a_3\cos(\omega T_s) + a_4\cos(2\omega T_s) + \cdots + a_k\cos((k-2)\omega T_s)\right] + \cdots \\ +2a_{k-2}\left[a_{k-1}\cos(\omega T_s) + a_k\cos(2\omega T_s)\right] + 2a_{k-1}a_k\cos(\omega T_s) \end{array}\right\}$$

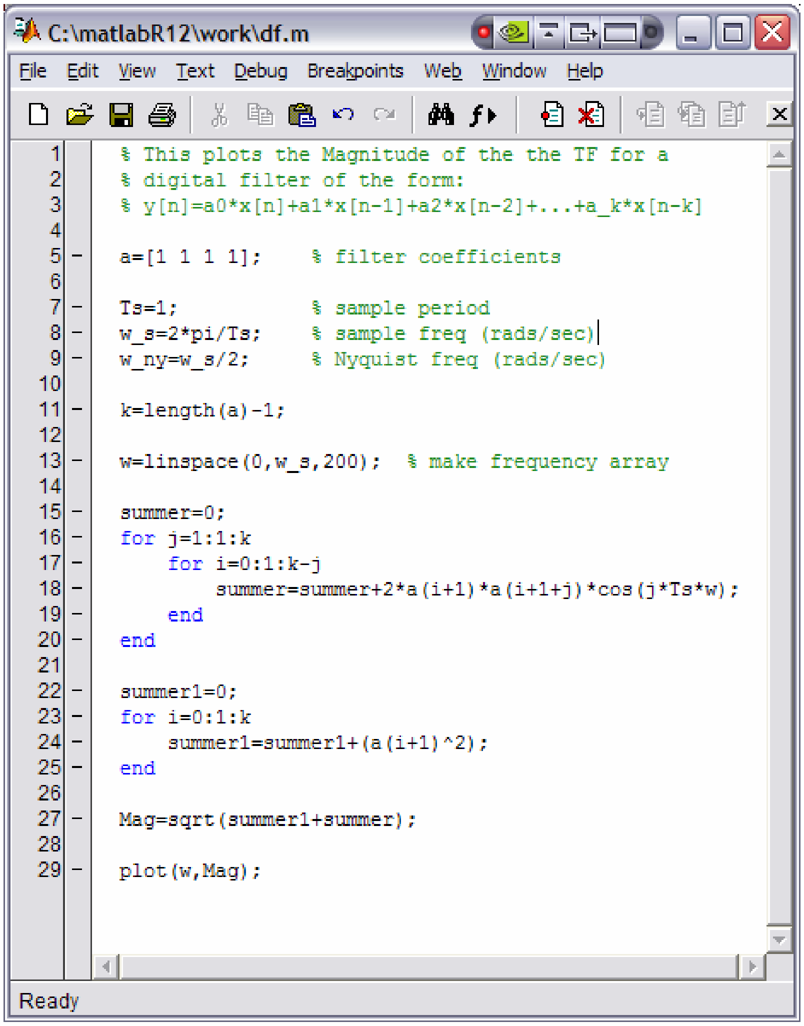
$$= sqrt\left\{\begin{array}{l} a_0^2 + a_1^2 + a_2^2 + \cdots + a_k^2 + \\ 2\left(a_0a_1 + a_1a_2 + a_2a_3 + \cdots + a_{k-1}a_k\right)\cos(\omega T_s) + \\ 2\left(a_0a_2 + a_1a_3 + a_2a_4 + \cdots + a_{k-2}a_k\right)\cos(2\omega T_s) + \\ 2\left(a_0a_3 + a_1a_4 + a_2a_5 + \cdots + a_{k-3}a_k\right)\cos(3\omega T_s) + \cdots \\ 2\left(a_0a_{k-1} + a_1a_k\right)\cos((k-1)\omega T_s) + 2a_0a_k\cos((k-1)\omega T_s) \end{array}\right\}$$

$$= sqrt\left\{\begin{array}{l} \sum_{i=0}^{k} a_i^2 + 2\cos(\omega T_s)\sum_{i=0}^{k-1} a_i a_{i+1} + 2\cos(2\omega T_s)\sum_{i=0}^{k-2} a_i a_{i+2} + \\ 2\cos(3\omega T_s)\sum_{i=0}^{k-3} a_i a_{i+3} + \cdots + 2\cos((k-1)\omega T_s)\sum_{i=0}^{1} a_i a_{i+k-1} + \\ 2\cos(k\omega T_s)a_0a_k \end{array}\right\}$$

Finally, this can be written as a double summation:

$$\boxed{\left|G(\omega)\right| = \left|\frac{y_n}{x_n}\right| = \sqrt{\sum_{i=0}^{k} a_i^2 + \sum_{j=1}^{k}\sum_{i=0}^{k-j} 2a_i a_{i+j}\cos(j\omega T_s)}} \qquad (1.2)$$

This can be implemented in MATLAB with the following script:



<u>Example</u>

Consider the simple derivative filter given by:

$$y_n = \frac{x_n - x_{n-1}}{T_s}$$

for which,

$$a_0 = \frac{1}{T_s}$$

and

$$a_1 = -\frac{1}{T_s}$$

Plugging into (1.2):

$$
\begin{aligned}
\left|G(\omega)\right| &= \sqrt{\sum_{i=0}^{1} a_i^2 + \sum_{j=1}^{1}\sum_{i=0}^{1-j} 2a_i a_{i+j}\cos\left(j\omega T_s\right)} \\
&= \sqrt{\left[\sum_{i=0}^{1} a_i^2 + \sum_{i=0}^{0} 2a_i a_{i+1}\cos\left(\omega T_s\right)\right]} \\
&= \sqrt{a_0^2 + a_1^2 + 2a_0 a_1 \cos\left(\omega T_s\right)} \\
&= \sqrt{\frac{2}{T_s^2} - \frac{2}{T_s^2}\cos\left(\omega T_s\right)} \\
&= \frac{1}{T_s}\sqrt{2 - 2\cos\left(\omega T_s\right)}
\end{aligned}
$$

which is the correct result. The plot of the TF is shown below: